Derivation of the Fraunhofer Diffraction Formula for the Point Spread Function of an Optical Microscope

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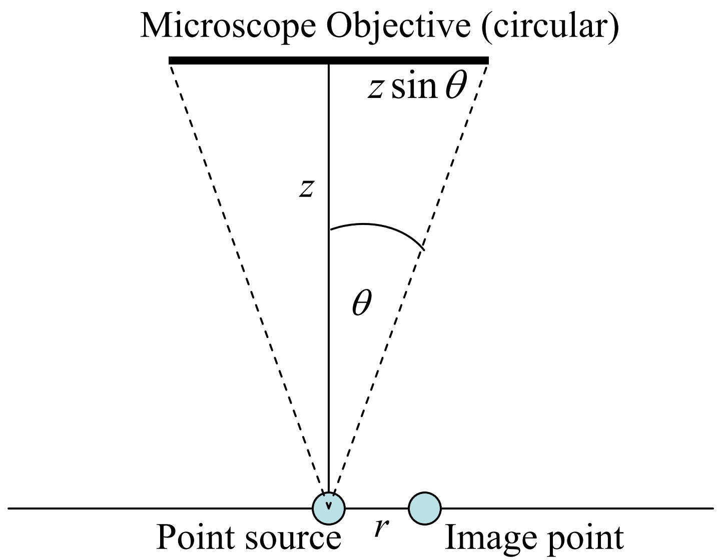
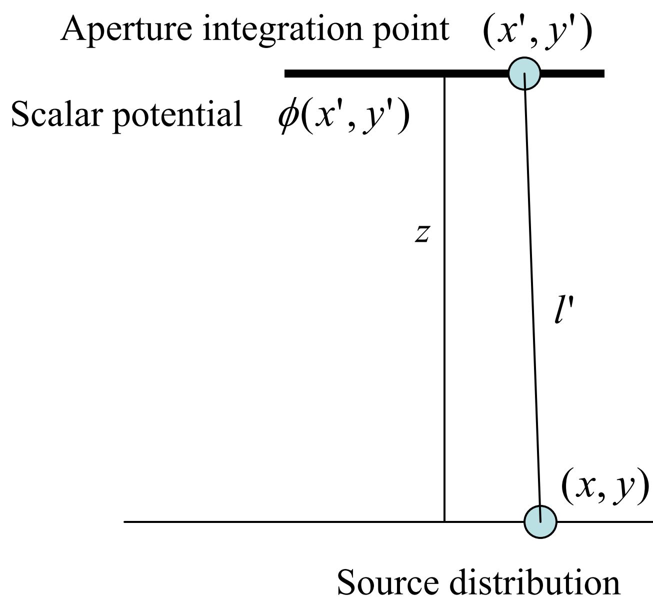

Microscope Objective (circular)
$z \sin\theta$
z
θ
Point source
r
Image point



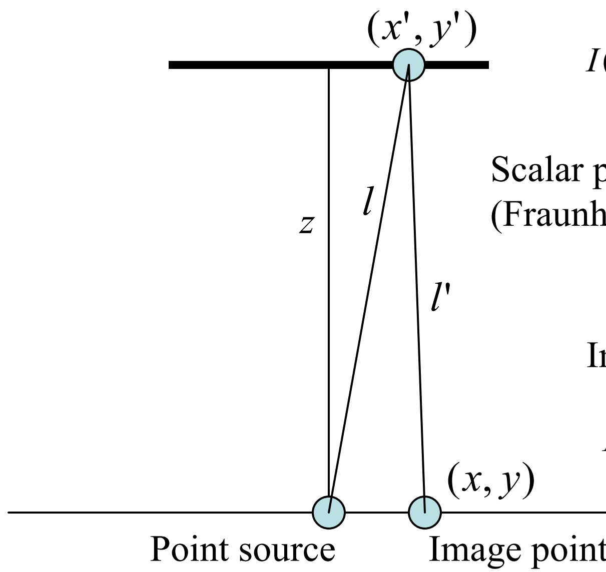
Imaging model:

$$I(x,y) = \left[\int_{aperture} e^{-ikl'} \phi(x', y') \, dx' \, dy' \right]^2$$

$$k = \frac{2\pi n}{\lambda}$$

n = index of refraction

λ = free space wavelength



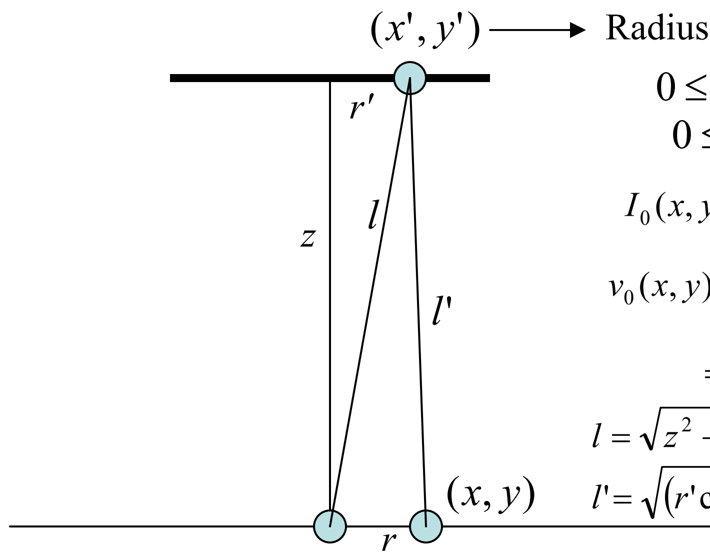
$$I(x,y) = \left[\int_{aperture} e^{-ikl'} \phi(x',y') dx' dy' \right]^2$$

Scalar potential from point source:
(Fraunhofer picture: ignore amplitude factor)

$$\phi_0(x',y') = e^{ikl}$$

Image from point source:

$$I_0(x,y) = \left[\int_{aperture} e^{-ikl'} e^{ikl} dx' dy' \right]^2$$



$$r = \sqrt{x^2 + y^2}$$

Radius r', polar angle ψ

$$0 \le r' \le z \sin\theta$$

$$0 \le \psi \le 2\pi$$

$$I_0(x, y) = \left[v_0(x, y)\right]^2$$

$$v_0(x, y) = \int_{aperture} e^{-ikl'} e^{ikl} dx' dy'$$

$$= \int_0^{z \sin\theta} \int_0^{2\pi} e^{ik(l-l')} r' d\psi dr'$$

$$l = \sqrt{z^2 + r'^2}$$

$$l' = \sqrt{\left(r' \cos\psi - r\right)^2 + r' \sin^2\psi + z^2}$$

Assume $r << z$

$$l - l' \approx \frac{-1}{2}\frac{r^2}{\sqrt{z^2 + r'^2}} + \frac{rr'\cos\psi}{\sqrt{z^2 + r'^2}}$$

$$v_0 = \int_0^{z\sin\theta} e^{-\frac{ik}{2}\frac{r^2}{\sqrt{z^2+r'^2}}} \int_0^{2\pi} e^{ik\frac{rr'\cos\psi}{\sqrt{z^2+r'^2}}} r'\,d\psi dr'$$

$$= \int_0^{z\sin\theta} e^{-\frac{ik}{2}\frac{r^2}{\sqrt{z^2+r'^2}}} \frac{2}{\pi} \mathrm{J}_0\left(\frac{krr'}{\sqrt{z^2 + r'^2}}\right) r'\,dr'$$

Again assume $r << z$

$$e^{-\frac{ik}{2}\frac{r^2}{\sqrt{z^2+r'^2}}} \approx 1$$

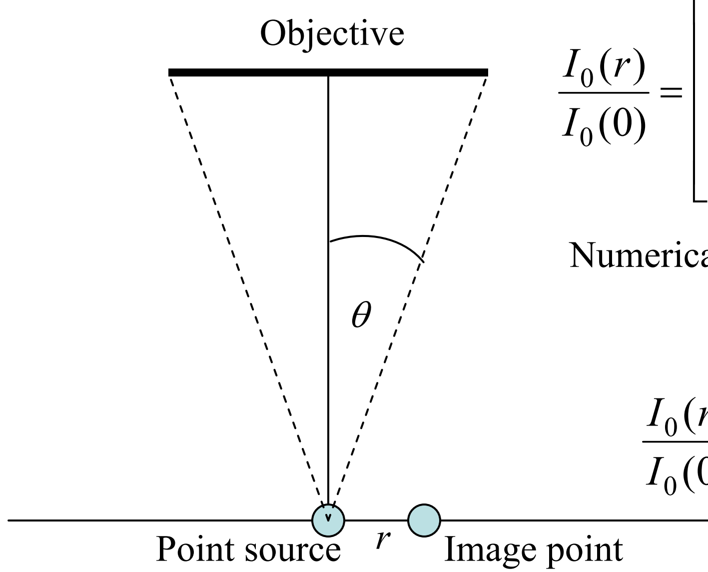
$$v_0 = \frac{2}{\pi}\int_0^{z\sin\theta} \mathrm{J}_0\left(\frac{krr'}{\sqrt{z^2+r'^2}}\right) r'\,dr'$$

$$= \frac{2}{\pi}\frac{z^2}{k^2r^2}\int_0^{kr\sin\theta} u\,\mathrm{J}_0(u)\,du$$

$$v_0(r) = \frac{2}{\pi}\frac{z^2}{k^2r^2}kr(\sin\theta)\mathrm{J}_1(kr\sin\theta)$$

$$I_0(r) = \left[\frac{2}{\pi} \frac{z^2}{k^2 r^2} kr(\sin\theta) \mathrm{J}_1(kr\sin\theta) \right]^2$$

$$\frac{I_0(r)}{I_0(0)} = \left[\frac{2\mathrm{J}_1\left(2\pi \frac{n\sin\theta}{\lambda} r\right)}{r} \right]^2$$

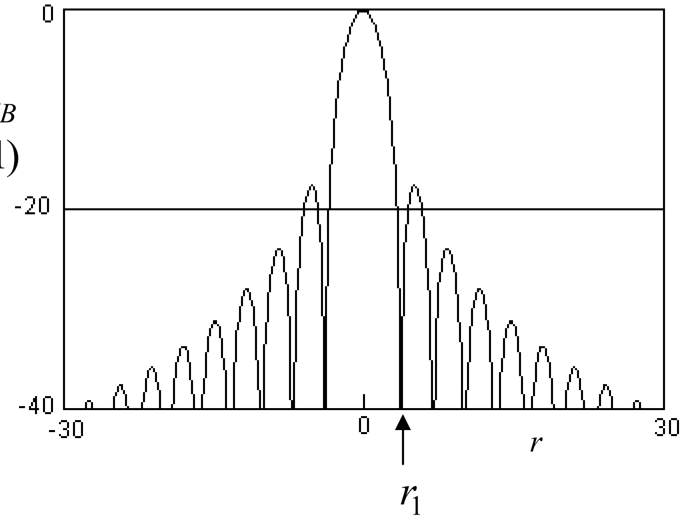


$$\frac{I_0(r)}{I_0(0)} = \left[\frac{2J_1\left(2\pi\frac{n\sin\theta}{\lambda}r\right)}{r}\right]^2$$

$$\text{Numerical Aperture } NA \equiv n\sin\theta$$

$$a \equiv 2\pi\frac{NA}{\lambda}$$

$$\frac{I_0(r)}{I_0(0)} = \left[\frac{2J_1(ar)}{r}\right]^2$$



psf_{dB} $(a=1)$

$$a \equiv 2\pi \frac{NA}{\lambda}$$

$$psf(r) = \frac{I_0(r)}{I_0(0)} = \left[\frac{2J_1(ar)}{r}\right]^2$$

$$psf_{dB} = 10\ \log_{10}(psf)$$

The first zero of $J_1(ar)$ is at $ar_1 = 3.84171$
The radius of the first minimum is Rayleigh's criterion for the resolution limit. Solving for r_1:

$$r_1 = 0.61\frac{\lambda}{NA}$$